Fuel-Optimal Maneuvers of a Spacecraft Relative to a Point in Circular Orbit

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Fuel-optimal maneuvers of a spacecraft relative to a body in circular orbit are investigated using a point mass model in which the magnitude of the thrust vector is bounded. This model assumes constant exhaust velocity, fixed flight time, and constant mass. An inverse square law gravitational force function is linearized about the center of the body in circular orbit and the resulting equations of motion describing the spacecraft are linear. The differential equations describing the Lawden primer vector can be solved and, in the planar case, the primer locus in general takes a cycloidal shape. In degenerate situations, elliptical, straight-line, or fixed-point shapes occur. Other shapes appear in the nonplanar cases. The nature of the fuel-optimal maneuvers is determined by the geometric relationship of the primer locus to the unit sphere. All nonsingular optimal maneuvers consist of intervals of full thrust and coast and are found to contain at most seven such intervals during one period of revolution. Only four cases where singular solutions occur are possible, two of which occur inside the orbital plane and two outside. Computer simulations of optimal flight path shapes and switching functions are depicted graphically for boundary conditions that represent various rendezvous problems. A simulation with seven thrust/coast phases in one period is presented.

Introduction

MOST of the immediate applications of space involve a body in circular orbit. Fuel-optimal maneuvers of a spacecraft relative to such a body constitute the subject of this investigation.

Much work has been done in the general area of fuel-optimal space maneuvers. Representative is that done by Lawden,[1] Leitmann,[2] Edelbaum,[3] and, more recently, Marec.[4] Other work can be found from the references cited in these and from the surveys done by Edelbaum,[5] Bell,[6] Robinson,[7] and Gobetz and Doll.[8]

In the present investigation, a coordinate system is centered in a satellite moving at constant speed in a circular orbit about a planet, the position and velocity of a spacecraft are considered relative to this coordinate system, and an inverse square law gravitational force function acting on the spacecraft is linearized about the center of the satellite. This model was used by Wheelon[9] in 1959 and by Clohessy and Wiltshire[10] in 1960. The coordinate system is set up in such a way that the positive x_1 axis is in the direction of the orbital velocity of the satellite, the positive x_2 axis is directed away from the center of the planet, and the positive x_3 axis completes a right-handed system. Except for the way the positive axes are defined, the present system is the same as that of Clohessy and Wiltshire.[10]

We assume the spacecraft to have a constant mass m concentrated at a point $[x_1(t), x_2(t), x_3(t)]$ relative to the above coordinate system and having a variable-thrust vector of bounded magnitude mb. The flight time t_f is assumed fixed. Denoting the angular speed of the satellite by Ω and the thrust acceleration vector relative to the above coordinate system by $b[u_1(t), u_2(t), u_3(t)]$, the spacecraft dynamics are represented by the following equations:

$$\ddot{x}_1 - 2\Omega\dot{x}_2 = bu_1(t)$$
$$\ddot{x}_2 + 2\Omega\dot{x}_1 - 3\Omega^2 x_2 = bu_2(t)$$
$$\ddot{x}_3 + \Omega^2 x_3 = bu_3(t)$$

where the dot signifies differentiation with respect to time t. These are identical to the equations determined by Clohessy and Wiltshire.[10] They will be used to define the differential equations (2) in the following section. For the case in which $u(t)$ is identically zero, these differential equations have been solved by both Wheelon[9] and Clohessy and Wiltshire.[10] This solution is of the same form as the solution of Lawden's primer equations for "coast" trajectories [Ref. 1, Chap. 5, Eqs. (5.45-5.47)] and also Eq. (14) in this paper, which is valid for either the "powered" or "coast" phases of an optimal trajectory.

Since we are assuming constant exhaust velocity, a minimum fuel maneuver is one in which the integral of the magnitude of the thrust is minimized.

It will be shown that for $\Omega \neq 0$ the nonsingular fuel-optimal maneuvers consist of intervals of full thrust and coast and that, at most, seven such intervals can occur for flight times less than or equal to $2\pi/\Omega$. This is because the Lawden primer vector can intersect the unit sphere at most six times during one period of revolution and the number of points of intersection determines the number of switches during an optimal maneuver. For the problem under consideration, the primer locus may take several geometric shapes, including helical types, cycloidal types, ellipses, and others. The helical and cycloidal types may intersect a unit sphere in as many as six points during one period, whereas the ellipses can intersect in at most four points. In the cases in which the primer locus is an ellipse, at most four switches are possible during one period of revolution. For $\Omega \neq 0$, it will also be shown that there are only four cases in which singular solutions exist and these are of only two distinct types, one occurring inside the orbital plane and the other outside. Finally, we present the results of the computer simulation of optimal maneuvers for various rendezvous problems. We will display a simulation in which seven thrust/coast phases occur during one period of revolution.

Some similarities should be noted between these results and those found in Marec (Ref. 4, Chaps. 6 and 7) in which he considers small-amplitude optimal transfers and rendezvous using orbital elements. He shows that for transfers between near-circular orbits the locus of the primer vector always becomes an ellipse where at most four switches are possible in a revolution. For the rendezvous case, he does not determine

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the number of switches. He shows also that only two types of singular solutions can occur. These are the two types that we find are defined by Eqs. (20) and (21).

Mathematical Analysis

The problem of minimum fuel maneuvers of a spacecraft about a body moving in a circular orbit is formulated as follows.

Let $\mathscr{U}$ denote the set of all Lebesgue measurable functions that map the real closed interval $[0, t_f]$ into the closed unit ball in R^3. We seek $u \in \mathscr{U}$ to minimize the functional

$$J[u] = \int_0^{t_f} |u(t)| \mathrm{d}t \tag{1}$$

where $|\ |$ denotes the Euclidean norm or magnitude, subject to the differential equations

$$\begin{aligned} \dot{x}(t) &= v(t) \\ \dot{v}(t) &= Ax(t) + Bv(t) + bu(t) \end{aligned} \tag{2}$$

which hold a.e. on $[0, t_f]$ and the boundary conditions

$$\begin{aligned} x(0) &= x_0 \qquad x(t_f) = x_f \\ v(0) &= v_0 \qquad v(t_f) = v_f \end{aligned} \tag{3}$$

where x_0, v_0, x_f, and v_f are specified points in R^3 and

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3\Omega^2 & 0 \\ 0 & 0 & -\Omega^2 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 2\Omega & 0 \\ -2\Omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4}$$

The scalar Ω is the orbital angular speed of the body in circular orbit and the scalar b is the maximum thrust magnitude divided by the mass of the spacecraft.

Since the integrand in Eq. (1) is independent of x and v, the Pontryagin minimum principle provides both necessary and sufficient conditions for a minimum (Ref. 11, Sec. 5.2). For boundary conditions in which the solution is normal, the Hamiltonian is

$$\begin{aligned} \mathscr{H} = \ell |u(t)| + p(t)^T v(t) + q(t)^T Ax(t) + q(t)^T Bv(t) \\ + bq(t)^T u(t) \end{aligned} \tag{5}$$

where the functions p and q mapping $[0, t_f]$ into R^3 are absolutely continuous solutions of the adjoint differential equations

$$\begin{aligned} \dot{p}(t) &= -A^T q(t) \\ \dot{q}(t) &= -B^T q(t) - p(t) \end{aligned} \tag{6}$$

and ℓ can be any positive real number. Upon setting $\ell = b$, we see that a nonsingular optimal control function is defined a.e. on $[0, t_f]$ by

$$\begin{aligned} u(t) &= 0, & |q(t)| &< 1 \\ u(t) &= -q(t)/|q(t)|, & |q(t)| &> 1 \end{aligned} \tag{7}$$

An optimal solution in which $|q(t)| = 1$ on a set of positive Lebesgue measure is called *singular* on that set.

Equation (7) shows that a nonsingular optimal control function has values that are either on the unit sphere or are zero (i.e., either full thrust or coast). Moreover, q determines entirely which of these situations occurs. The function q is called the Lawden primer vector (Ref. 1, Chap. 3). If $q(t)$ is outside of the unit ball, a full thrust is required, whereas a coast is required if $q(t)$ is inside the unit ball. Qualitative information about the nature of optimal solutions, such as the number of thrusts and coasts or whether or not the solution is singular, can be determined from the primer q. We shall therefore solve Eq. (6) for q and examine the various forms that this solution can take.

Eliminating $p(t)$ in Eq. (6) we obtain the differential equation

$$\ddot{q}(t) + B^T \dot{q}(t) - A^T q(t) = 0 \tag{8}$$

Before finding the general solution of Eq. (8), we note that $A^T = A$ and $B^T = -B$, so that if we eliminate v in Eq. (2) we obtain

$$\ddot{x}(t) + B^T \dot{x}(t) - A^T x(t) = bu(t) \tag{9}$$

During a coast interval in which $u(t) = 0$, the differential equations (8) and (9) are identical. For this reason $q(t)$ may be visualized as a point in orbit relative to a body moving in a circular orbit.

Using subscripts to denote the components of the vector $q(t)$, Eq. (8) may be replaced by the following system of scalar equations:

$$\begin{aligned} \ddot{q}_1(t) - 2\Omega \dot{q}_2(t) &= 0 \\ \ddot{q}_2(t) + 2\Omega \dot{q}_1(t) - 3\Omega^2 q_2(t) &= 0 \\ \ddot{q}_3(t) + \Omega^2 q_3(t) &= 0 \end{aligned} \tag{10}$$

Equation (7) shows that a boundary point t between a thrusting and coasting regime of $u(t)$ must satisfy the condition

$$|q(t)| = 1 \tag{11}$$

The function $|q(t)| - 1$ is called the *switching function* of $u(t)$. A value of t at which the switching function changes sign is called a *switch* of $u(t)$. Whenever a switch occurs, $u(t)$ changes from thrust to coast or from coast to thrust and conversely. Whenever a switch of $u(t)$ occurs, the solution of Eq. (10) intersects the unit sphere. Clearly, the number of switches cannot exceed the number of intersection points.

The form of the solution of Eq. (10) depends on whether or not $\Omega = 0$. The case $\Omega = 0$ corresponds to a problem of optimal maneuvers in deep space rather than about a circular orbit. In deep space, Eq. (8) becomes

$$\ddot{q}(t) = 0 \tag{12}$$

and its general solution is

$$q(t) = -p_0 t + q_0 \qquad (0 \le t \le t_f) \tag{13}$$

where p_0 and q_0 are constants in R^3. Geometrically, this solution describes a straight-line segment in R^3. In view of Eq. (11), this shows that an optimal maneuver in deep space can have at most two thrust intervals and one coast interval. This fact is well known and a discussion of this problem can be found in the literature.[1,3,4] It is known that the optimal control function is nonsingular for this problem, except in the case where the vectors p_0 and q_0 are colinear. In this case the relative trajectory degenerates to a straight line and the problem becomes the type treated by Athans and Falb (Ref. 12, Chap. 8) where a discussion of the singular solutions may also be found. A detailed discussion of the deep space problem can be found in Ref. 13.

There is more variety in the type of optimal maneuvers that can occur about a body in circular orbit. In the case where

$\Omega \neq 0$, the general solution of Eq. (10) is

$$q_1(t) = 2\rho\sin(\Omega t + \psi) - 3c_1 t + c_2$$
$$q_2(t) = \rho\cos(\Omega t + \psi) - 2c_1/\Omega \qquad (0 \le t \le t_f)$$
$$q_3(t) = \alpha\sin(\Omega t + \psi) + \beta\cos(\Omega t + \psi) \tag{14}$$

These equations and also Eq. (10) were found by Lawden [Ref. 1, Chap. 5, Eqs. (5.42-5.47)] in slightly different form, but they were derived only for "coast" trajectories. For that reason, they could not be used to determine when to switch during a thrust phase, as they can for our problem. We have shown in this case that Lawden's original results can be extended to a linearized model.

Geometrically the locus of Lawden's primer vector, as defined by Eq. (14), resembles a helical segment if ρ, c_1, and either α or β is nonzero. Less general situations can occur and will be considered here. Of special interest is the case in which $\alpha = \beta = 0$. In this case, the solution curve from Eq. (14) is confined to the plane of the circular orbit.

Solutions in the Orbital Plane

If the third component of each of the vectors x_0, v_0, x_f, and v_f in Eq. (3) is zero, then the solutions $u(t)$, $q(t)$, and $x(t)$ from Eqs. (7-9) are each contained in the orbital plane.

In this case the plane curve obtained from Eq. (14) resembles a type of cycloid if c_1 and ρ are nonzero. The relationship to a cycloid can be demonstrated by the following transformation:

$$\gamma_1 = -q_1/2 \qquad \gamma_2 = -q_2 \tag{15}$$

The curve defined by $\gamma = (\gamma_1, \gamma_2)$ is a type of cycloid. If we introduce the following change of variables:

$$\theta = \Omega t + \psi$$
$$\lambda = 3c_1/(2\Omega)$$
$$c = -3c_1\psi/(2\Omega) - c_2/2 \tag{16}$$

the curve takes the standard form

$$\gamma_1 = \lambda\theta - \rho\sin\theta + c$$
$$\gamma_2 = \lambda - \rho\cos\theta + \lambda/3 \tag{17}$$

This curve is a cycloid if $\rho = |\lambda|$ and $\rho > 0$, a prolate cycloid if $\rho > |\lambda| > 0$, and a curtate cycloid if $0 < \rho < |\lambda|$. In each case, the curve is periodic with a period of 2π. For this reason, the curve defined by q is a segment of a periodic curve having the orbital period $2\pi/\Omega$. It has periodic loops if and only if $\rho > 3|c_1|/(2\Omega) > 0$.

The nature of the optimal thrusting sequence is determined by the relationship of $q(t)$ to the unit circle in the orbital plane. This is depicted in Fig. 1 for the case where $\Omega = 0$. Since a straight line can intersect a circle in at most two points, it is seen that a fuel-optimal maneuver in deep space can have at most two switches. If $\Omega \neq 0$, more switches can occur as is demonstrated in Fig. 2 were $q(t)$ takes a cycloidal shape. There is an essential difference between the deep space case and the circular orbit case even if t_f is small. This is because, even though θ may be restricted to an arbitrarily small interval, there are values of the constants λ, ρ, and c so that a prolate cycloid can have a complete loop on that interval. For this reason, even if t_f is small, there are values of ρ, c_1, and c_2 such that $q(t)$ can intersect the unit circle more than twice.

Equation (15) transforms the unit circle into the ellipse

$$\gamma_1^2 + 4\gamma_2^2 = 4 \tag{18}$$

This ellipse can intersect the curve defined by Eq. (17) at most six times in one period and more than six times over larger intervals than a period. This indicates that at most six switches are possible during one period of a fuel-optimal maneuver about a body in a circular orbit and more are possible for longer flight times.

A case where six switches occur is presented in Fig. 3 where the curve $q(t)$ intersects the unit circle at six points. This corresponds to a "thrust/coast/thrust/coast/thrust/coast/thrust" maneuver, a thrusting sequence consisting of seven phases. This information can be seen more easily from Fig. 4, which presents the magnitude of q vs the orbital angle θ defined in Eq. (16). The thrust intervals occur where $|q|$ is greater than one and the coast intervals occur where $|q|$ is less than one. A seven-phase maneuver can also occur for certain smaller flight times if ρ, c_1, and c_2 are chosen correctly.

In the degenerate cases, if $c_1 = 0$ and $\rho \neq 0$, then $q(t)$ defines a segment of an ellipse whose ratio of major-to-minor axis is two. An ellipse can intersect the unit circle in at most four points, so the maximum number of switches that can occur in an optimal maneuver in this case is four in one period —although more can occur for flight times exceeding one period.

If $\rho = 0$ and $c_1 \neq 0$, then $q(t)$ defines a straight-line segment as in the deep space situation and at most two switches are possible.

If both c_1 and ρ are zero, no switches can occur. This is the only situation in the orbital plane case where singular solu-

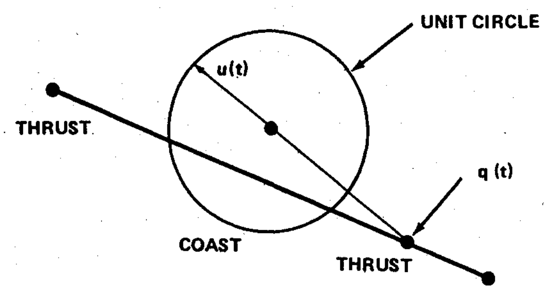

Fig. 1 Nature of fuel-optimal thrusting sequence in deep space.

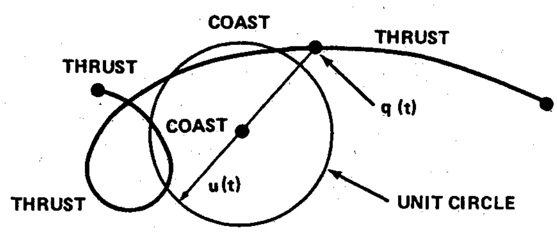

Fig. 2 Nature of fuel-optimal thrusting sequence about a point moving in circular orbit.

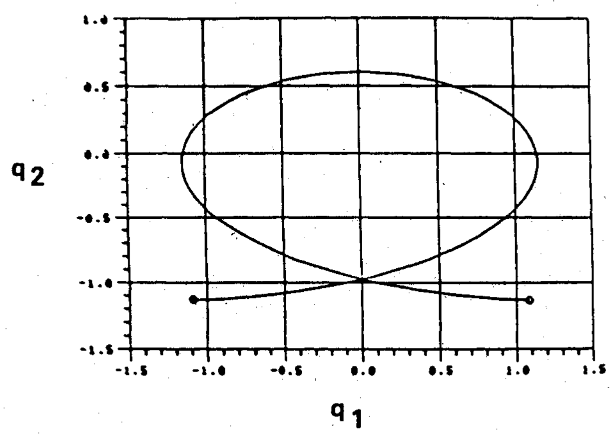

Fig. 3 Seven-phase optimal thrusting sequence in one period.

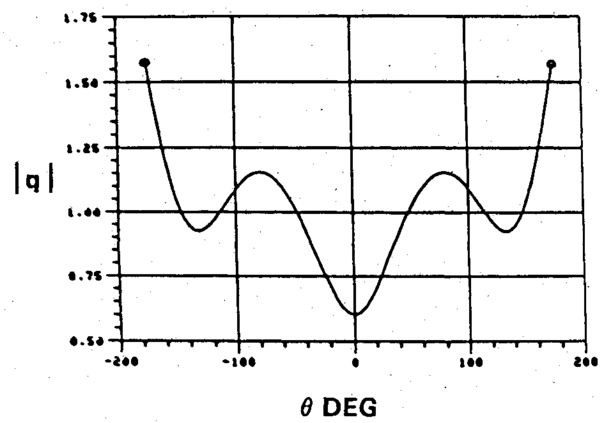

Fig. 4 Magnitude of q over one period for the seven-phase optimal thrusting sequence.

tions can occur. This happens if $c_2 = \pm 1$. This fact will be shown below.

Solutions out of the Orbital Plane

Cases where $q(t)$ as defined by Eq. (14) is not in the orbital plane will be examined in this subsection. First, consider the cases where $c_1 \neq 0$. Singular solutions are impossible for these cases because $c_1 \neq 0$ in Eq. (14) shows that Eq. (11) cannot be satisfied identically.

If α and β are not both zero and if ρ is nonzero, then the solution of Eq. (14) defines a helical-type path that intersects the unit sphere on at most six points during one period. This establishes at most six switches of $u(t)$ during one period of an optimal maneuver.

If $\rho = 0$ and α and β are not both zero, then Eq. (14) defines a shifted sine curve. This curve can intersect the unit sphere at six points but not more. Also, in this situation, an optimal maneuver has at most six switches in one period.

For cases in which $c_1 = 0$, singular solutions occur if and only if Eq. (11) is satisfied identically, that is

$$\beta^2 - \alpha^2 = 3\rho^2, \quad \alpha\beta = 0, \quad c_2\rho = 0, \quad \rho^2 + \beta^2 + c_2^2 = 1 \tag{19}$$

Equation (19) is satisfied only in the following cases:

$$\rho = \tfrac{1}{2}, \quad c_2 = 0, \quad \alpha = 0, \quad \beta = \pm\sqrt{3}/2 \tag{20}$$

$$\rho = 0, \quad c_2 = \pm 1, \quad \alpha = 0, \quad \beta = 0 \tag{21}$$

These are the only cases in which singular solutions occur. Equation (20) provides the only cases in which singular solutions exist outside of the orbital plane and Eq. (21) establishes the existence of the singular solutions in the orbital plane previously mentioned.

In the case where $\rho \neq 0$, Eq. (14) defines, in general, an ellipse that either intersects the unit sphere in at most four points or that is identically on the unit sphere. The latter is the singular situation described by Eq. (20). The former indicates

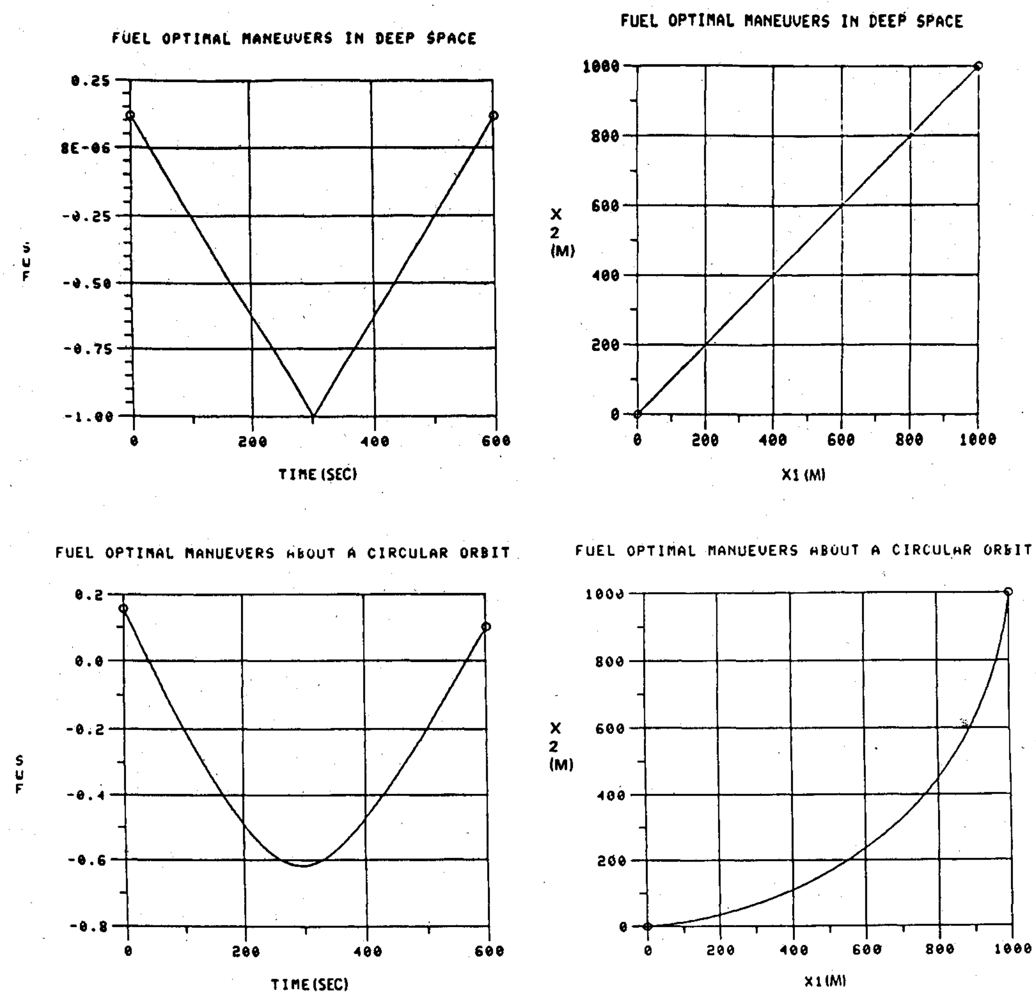

Fig. 5 Comparison of switching functions and flight paths for deep space and 435 km altitude circular orbit: $X(0) = (1000, 1000)$, $V(0) = (0, 0)$, $X(TF) = (0, 0)$, $V(TF) = (0, 0)$.

that at most four switches in one period are possible for a nonsingular optimal maneuver in this case.

If $\rho = 0$ and α and β are not both zero, then Eq. (14) defines a point moving with simple harmonic motion along a line segment. This moving point crosses the unit sphere at no more than four values of t. In this case as well, no more than four switches can occur during one period of an optimal maneuver.

Summary of Results

Several different forms of solutions to Eq. (10) can occur. If $\Omega = 0$ the solution is a line segment and the optimal control consists of intervals of full thrust and coast with at most two switches. If $\Omega \neq 0$, the solution may be a helical-type segment, a shifted sine curve segment, a horizontal straight-line segment, a prolate or curtate cycloidal-type segment, or a segment of an ellipse. The optimal control consists of intervals of full thrust and coast in each of these cases. More than two switches are possible even for small flight times. If $c_l \neq 0$, there is a maximum of six switches possible in one period. If $c_l = 0$, the maximum number of switches that can occur in one period is four.

If $\Omega \neq 0$, only two cases exist in which singular solutions [defined by Eq. (20)] can occur outside of the orbital plane. Also, only two cases where singular solutions [defined by Eq. (21)] exist can occur on the orbital plane.

Computer Simulations

This section presents the flight path and switching function for several fuel-optimal maneuvers in a plane, both in deep space and about a body in circular orbit. These results were obtained by solution of a two-point boundary-value problem using a program developed at the NASA George C. Marshall Space Flight Center.

Figures 5 and 6 compare an optimal soft rendezvous with a body fixed in deep space and an optimal soft rendezvous with a body in circular orbit for identical initial conditions. These simulations were based on a spacecraft mass of approximately 3400 kg, maximum thrust of 267 N, and a flight time of 600 s. The upper part of the figures illustrates the optimal switching function and flight path shape for the deep space rendezvous ($\Omega = 0$ rad/s), whereas the lower part presents this information for rendezvous with a body moving in a 435 km altitude circular orbit ($\Omega = 0.001122$ rad/s). The similarities and differences in shape are apparent.

Figure 7 presents an optimal soft rendezvous of a spacecraft in the 435 km altitude circular orbit with a body 1 km behind in the same orbit for flight times of 600 s, one-half period (2800 s), and three-fourths period (4200 s). The mass and maximum thrust are the same as in the previous case.

Figure 8 also presents a soft rendezvous in which the same spacecraft is ferrying a mass nine times heavier than itself to a body in the 435 km circular orbit. The spacecraft begins 1 km

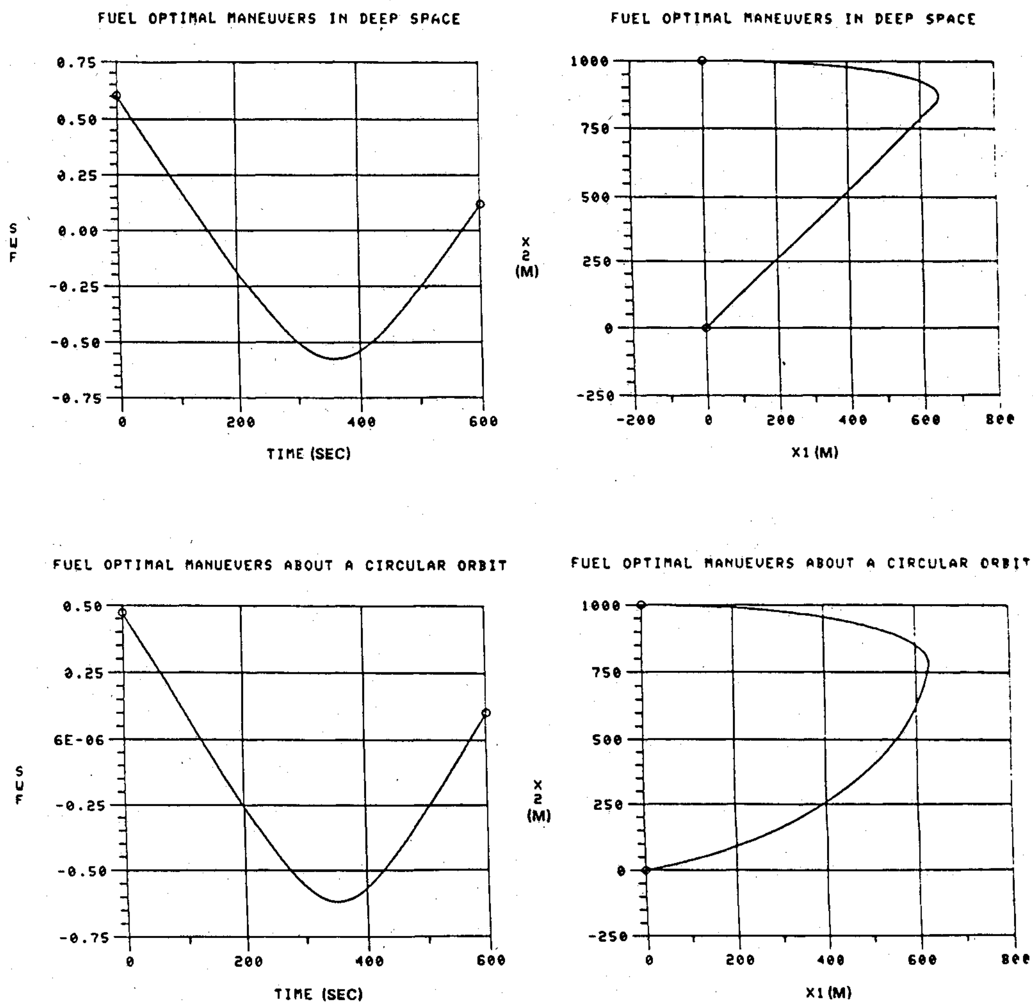

Fig. 6 Comparison of switching functions and flight paths for deep space and 435 km altitude circular orbit: $X(0) = (0, 1000)$, $V(0) = (10, 0)$, $X(TF) = (0, 0)$, $V(TF) = (0, 0)$.

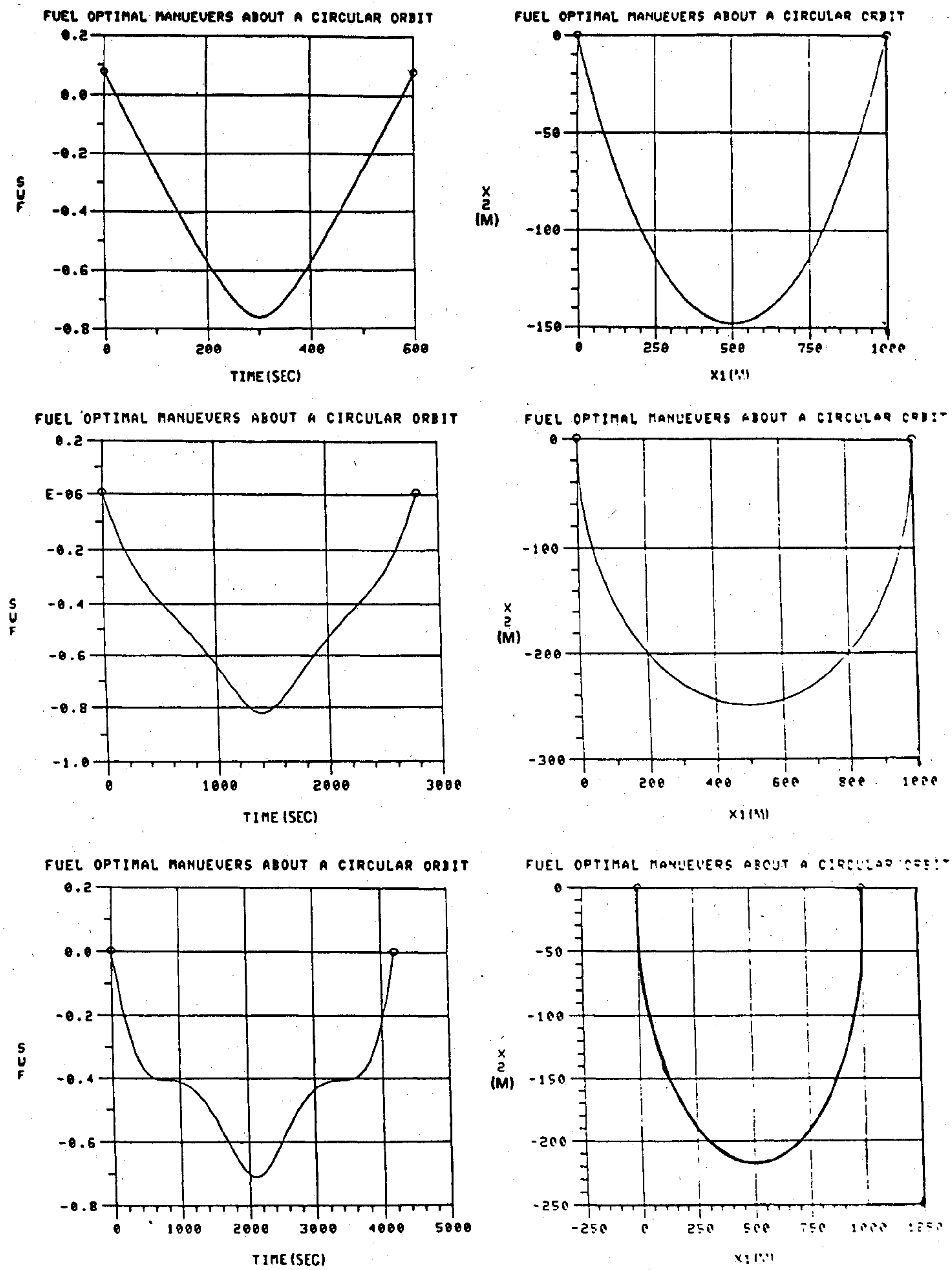

Fig. 7 Switching function and flight path for different flight times for 435 km circular orbit: $X(0) = (1000, 0)$, $V(0) = (0, 0)$, $X(\mathrm{TF}) = (0, 0)$, $V(\mathrm{TF}) = (0, 0)$.

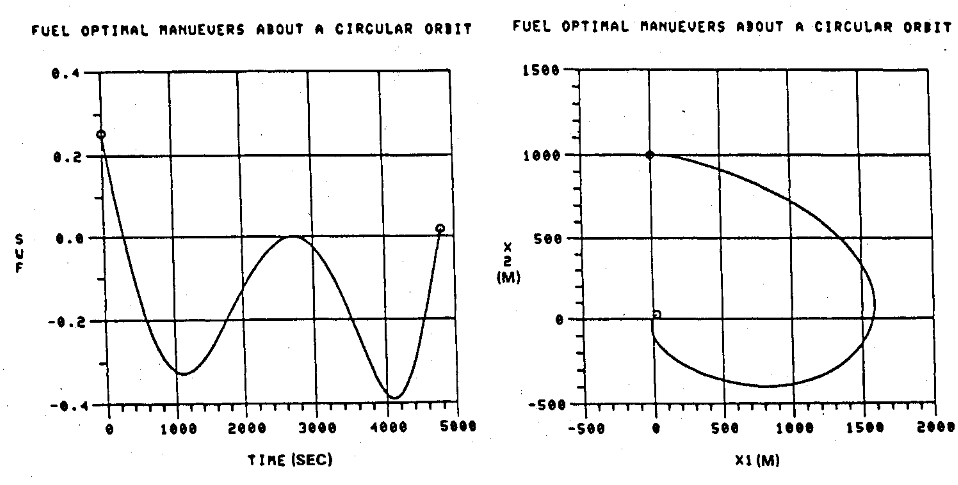

Fig. 8 Insertion of mass from elliptical into 435 km circular orbit: $X(0) = (0, 1000)$, $V(0) = (0, 0)$, $X(\mathrm{TF}) = (0, 0)$, $V(\mathrm{TF}) = (0, 0)$, optimal maneuver requires midcourse thrust.

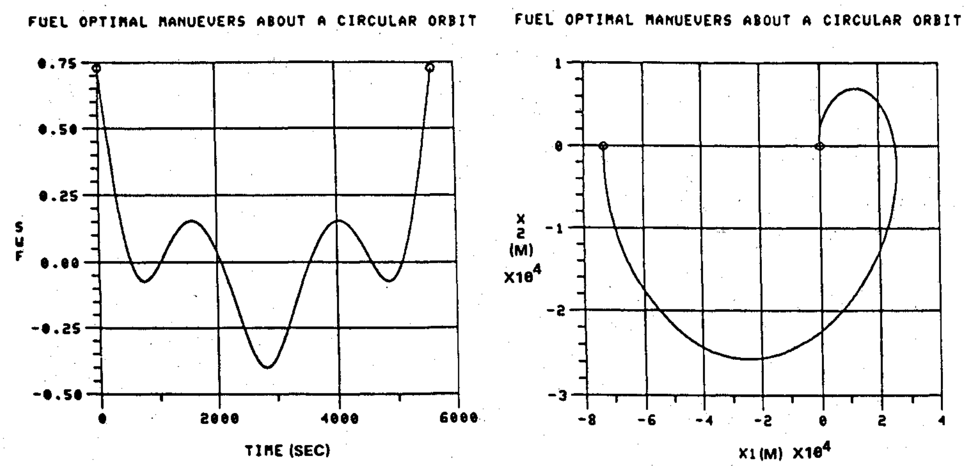

Fig. 9 Optimal maneuver with four thrusts and three coasts in one period (5600 s): $X(0) = (0,0)$, $V(0) = (0,0)$, $X(TF) = (-73{,}000, 0)$, $V(TF) = (0,35)$.

above the body with the same velocity. The flight time is long, six-sevenths of a period (4800 s) and the optimal thrusting function has four switches. The thrusting sequence is "thrust/coast/thrust/coast/thrust."

Figure 9 depicts a situation in which the spacecraft pushes a mass nine times heavier than itself out of circular orbit to a point in an elliptical orbit 73 km behind. This optimal maneuver has flight time of one period (5600 s) and the optimal switching function is determined from the curve $q(t)$ presented in Fig 3. This can be seen from the similarity between Fig. 4 and the switching function shown in Fig. 9. This is an example of an optimal maneuver that has a thrusting sequence requiring the maximum of seven phases in one period.

Conclusions

The gravitational force acting on a spacecraft near a body in circular orbit can be linearized about the center of the body in circular orbit and the resulting equations of motion of the spacecraft relative to the orbiting body are linear. Using these as the state variable equations, a fuel-optimal rendezvous problem is formulated, and Lawden's primer equations for "coast" trajectories hold for this model.

The locus of the Lawden primer can take several types of shapes for this problem. In the orbital plane its nondegenerate shape is cycloidal, but it can also take the shape of an elliptical segment, a straight-line segment, or a fixed point. The latter defines the only singular case occurring inside the orbital plane. Outside the orbital plane, the primer takes a helical shape in the nondegenerate situation. Other shapes include segments of shifted sine curves or ellipses. The only situation in which singular solutions can appear outside of the orbital plane is the case where the primer locus is circular, its plane inclined ± 60 deg with respect to the orbital plane.

Whether the primer vector is contained in the orbital plane or not, an optimal maneuver may consist of as many as seven thrust/coast phases during one period of revolution but not more. For any shorter interval of time, there are boundary values where a maneuver with four to seven phases can occur.

For this model the computational problems are not unusually difficult. Computer simulations were presented for a variety of planar cases. Simulations having both five and seven thrust/coast phases are included. It was found generally that for small flight times and boundary conditions of small magnitudes, the optimal maneuvers displayed no more than three thrust/coast phases.

Since many of the space missions now under consideration involve objects such as space stations that must be maintained in a prescribed circular orbit about the Earth and are also serviced by smaller spacecraft, this work should have practical applications.

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